A Conclusive Test of Abelian Dominance Hypothesis for Topological Charge in the QCD Vacuum[*]

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We study the topological feature in the QCD vacuum based on the hypothesis of abelian dominance. The topological charge $Q_{\mathrm{SU(2)}}$ can be explicitly represented in terms of the monopole current in the abelian dominated system. To appreciate its justification, we directly measure the corresponding topological charge Q_{Mono}, which is reconstructed only from the monopole current and the abelian component of gauge fields, by using the Monte Carlo simulation on SU(2) lattice. We find that there exists a one-to-one correspondence between $Q_{\mathrm{SU(2)}}$ and Q_{Mono} in the maximally abelian gauge. Furthermore, Q_{Mono} is classified by approximately discrete values.

A stimulating idea of the abelian gauge fixing was proposed by 't Hooft [1] and also independently by Ezawa-Iwazaki [2]. After performing the partial gauge fixing to remain the abelian gauge degrees of freedom, one knows that the non-abelian gauge theory is reduced to the abelian gauge theory with magnetic monopoles. Such topological excitations would play an essential role on color confinement [3]. Furthermore, Ezawa and Iwazaki stressed the hypothesis of abelian dominance [2]:

- Only the abelian component of gauge fields is relevant at a long-distance scale.

- The non-abelian effects are mostly inherited by magnetic monopoles.

They then demonstrated monopole condensation in a long-distance scale based on an argument about the "energy-entropy balance" on the monopole current [2].

The recent lattice QCD simulations show several evidences of monopole condensation [3]. This means that confinement could be regarded as the dual version of the superconductivity. Second important result from the lattice QCD simulation indicates that abelian dominance for some physical quantities, *e.g.* the string tension [3] and the chiral condensate [4], is actually realized in the maximally abelian (MA) gauge. In this gauge, at least, the abelian component could be the important dynamical degrees of freedom at a long-distance scale. Here, an unavoidable question arises. *In the abelian dominated system, is it possible for the non-abelian topological nature to survive?* For such a question, Ezawa and Iwazaki have proposed a remarkable conjecture [2]: once abelian dominance is postulated, the topological feature is preserved by the presence of monopoles.

Following the previous study of the fermionic zero-mode [5], we will show that the topological charge is explicitly represented in terms of monopoles in the lattice formalism by using the hypothesis of abelian dominance. Finally, we intend to numerically confirm its justification by means of the Monte Carlo simulation in the MA gauge. For simplicity, we restrict the discussion to the case of the SU(2) gauge group hereafter.

We first address the definition of the topological charge in the lattice formalism. The naive and field-theoretical definition of the topological density is given by

$$q(s) \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left\{P_{\mu\nu}(s)P_{\rho\sigma}(s)\right\} , \qquad (1)$$

where the clover averaged SU(2) plaquette is defined as $P_{\mu\nu}(s) \equiv \frac{1}{4}(U_{\mu\nu}(s) + U^{\dagger}_{-\mu\nu}(s) +$

$U_{\mu-\nu}^\dagger(s) + U_{-\mu-\nu}(s)$) with the convenient notation for the SU(2) link variable as $U_{-\mu}(s) = U_\mu^\dagger(s-\hat\mu)$. The topological charge Q_{cont} is naively extracted from the summation of the previous defined topological density over all site as leading order in powers of the lattice spacing a:

$$Q_L = -\frac{1}{16\pi^2}\sum_s q(s) \simeq Q_{\text{cont}} + \mathcal{O}(a^6) , \qquad (2)$$

where $Q_{\text{cont}} = \frac{1}{16\pi^2}\sum_s \text{tr}\left\{a^4 g^2 G_{\mu\nu}(s)^* G_{\mu\nu}(s)\right\}$.

Here, the SU(2) link variable is expected to be U(1)-like as $U_\mu(s) \simeq u_\mu(s) \equiv \exp\{i\sigma_3\theta_\mu(s)\}$, provided that the QCD vacuum is described as the abelian dominated system in a suitable abelian gauge. The angular variable; $\theta_\mu(s) \equiv \arctan[U_\mu^3(s)/U_\mu^0(s)]$ is defined in the compact domain $[-\pi,\pi)$. In the abelian dominated system, we might consider the abelian analog of the topological density [7] in stead of eq.(1) as

$$
\begin{aligned}
q_{\text{Abel}}(s) &= \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\text{tr}\left\{p_{\mu\nu}(s)p_{\rho\sigma}(s)\right\} \\
&= -\varepsilon_{\mu\nu\rho\sigma}\mathcal{S}_{\mu\nu}(s)\mathcal{S}_{\rho\sigma}(s) ,
\end{aligned} \qquad (3)
$$

where $p_{\mu\nu}(s) \equiv \frac{1}{4}\sum_{i,j=0}^1 u_{\mu\nu}(s-i\hat\mu-j\hat\nu)$ and $\mathcal{S}_{\mu\nu}(s) = \frac{1}{4}\sum_{i,j=0}^1 \sin[\theta_{\mu\nu}(s-i\hat\mu-j\hat\nu)]$.

Our next aim is to discuss the expression of the abelian analog of the topological density in the naive continuum limit $a \to 0$. This is because we need only the leading order term in powers of the lattice spacing to determine the corresponding topological charge. Here, one may notice that the U(1) elementary plaquette $u_{\mu\nu}$ is a multiple valued function as the U(1) plaquette angle; $\theta_{\mu\nu}(s) \equiv \partial_\mu\theta_\nu(s) - \partial_\nu\theta_\mu(s)$. Hence, we divide into $\theta_{\mu\nu}$ into two parts as

$$\theta_{\mu\nu}(s) = \bar\theta_{\mu\nu}(s) + 2\pi n_{\mu\nu}(s) , \qquad (4)$$

where $\bar\theta_{\mu\nu}$ is defined in the principal domain $[-\pi,\pi)$, which corresponds to the U(1) field strength. $n_{\mu\nu}$ can take the restricted integer; $0,\pm1,\pm2$. Taking the limit $a \to 0$, i.e. $\bar\theta_{\mu\nu} \to 0$, we thus arrive at the following expression:

$$q_{\text{Abel}}(s) \approx -\varepsilon_{\mu\nu\rho\sigma}\bar\Theta_{\mu\nu}(s)\bar\Theta_{\rho\sigma}(s) , \qquad (5)$$

which is rewritten in terms of the U(1) field strength; $\bar\Theta_{\mu\nu} \equiv \frac{1}{4}\sum_{i,j=0}^1 \bar\theta_{\mu\nu}(s-i\hat\mu-j\hat\nu)$.

Next, we intend to explicitly represent the topological charge in terms of monopoles. For the identification of monopoles, we follow DeGrand-Toussaint's definition in the compact U(1) gauge theory [6]. The monopole current is given by

$$k_\mu(s) \equiv \frac{1}{4\pi}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\bar\theta_{\rho\sigma}(s+\hat\mu) , \qquad (6)$$

which denotes the integer-valued magnetic current, because of the Bianchi identity on the U(1) plaquette angle; $\epsilon_{\mu\nu\rho\sigma}\partial_\nu\theta_{\rho\sigma} = 0$ [6].

To show the explicit contribution of monopoles to the topological charge, we introduce the dual potential $\mathcal{B}_\mu$ satisfying the following equation [8]:

$$\left(\Delta^2\delta_{\mu\nu} - \Delta_\mu\Delta_\nu\right)\mathcal{B}_\nu(s) = -2\pi\mathcal{K}_\mu(s) , \qquad (7)$$

where $\mathcal{K}_\mu(s) \equiv \frac{1}{8}\sum_{i,j,k=0}^1 k_\mu(s-i\hat\nu-j\hat\rho-k\hat\sigma)$ and Δ_μ denotes the nearest-neighbor central difference operator. It is worth mentioning that $\mathcal{K}_\mu$ satisfies the conservation law; $\Delta_\mu\mathcal{K}_\mu(s) = 0$. We can perform the Hodge decomposition on $\bar\Theta_{\mu\nu}$ with the dual potential $\mathcal{B}_\mu$ as

$$\bar\Theta_{\mu\nu}(s) = \Delta_\mu\mathcal{A}_\nu'(s) - \Delta_\nu\mathcal{A}_\mu'(s) + \varepsilon_{\mu\nu\rho\sigma}\Delta_\rho\mathcal{B}_\sigma(s), (8)$$

where $\mathcal{A}_\mu'$ is the Gaussian fluctuation [8]. After a little algebra, we find the explicit contribution of monopoles to the r.h.s of eq.5 as

$$\varepsilon_{\mu\nu\rho\sigma}\bar\Theta_{\mu\nu}(s)\bar\Theta_{\rho\sigma}(s) = 16\pi\mathcal{A}_\mu'(s)\mathcal{K}_\mu(s) + \cdots , \ (9)$$

where the ellipsis stands for the total divergence, which will drop in the summation over all site. Consequently, we arrive at the conjecture that *the topological feature is preserved by the presence of monopoles in the abelian dominated system*:

$$Q_{\text{cont}} \simeq -\frac{1}{16\pi^2}\sum_s q_{\text{Mono}}(s) , \qquad (10)$$

where $q_{\text{Mono}}(s) \equiv -16\pi\mathcal{A}_\mu'(s)\mathcal{K}_\mu(s)$.

Finally, we investigate above conjecture to justify by means of the Monte Carlo simulation. We generate the gauge configurations by using the standard SU(2) Wilson action on a 16^4 lattice at $\beta = 2.4$. First, we get the smoothed gauge configurations, which are eliminated undesirable fluctuations from the given Monte Carlo configurations through the naive cooling procedure. The realization of abelian dominance is established in
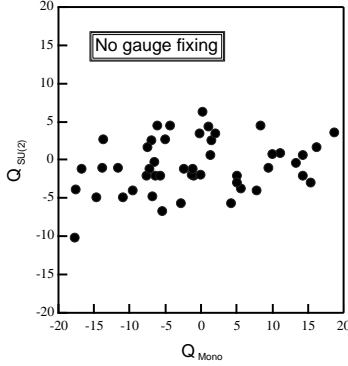
Figure 1(a): Scatter plot of $Q_{\mathrm{SU(2)}}$ vs. Q_{Mono} in the case of no gauge fixing.
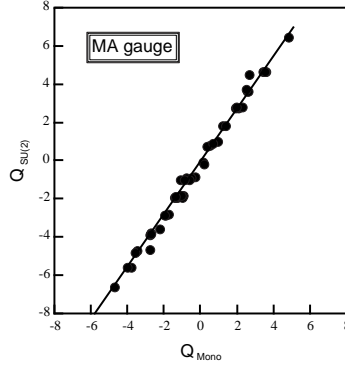
Figure 1(b): Scatter plot of $Q_{\mathrm{SU(2)}}$ vs. Q_{Mono} in the maximally abelian (MA) gauge.
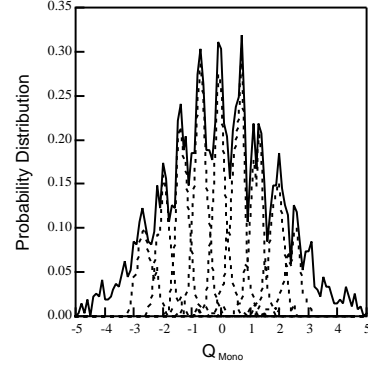
Figure 2: Probability distribution of Q_{Mono} in the maximally abelian (MA) gauge.

the MA gauge [3]. We then apply the smoothed gauge configurations after several cooling sweeps to the gauge transformation in order to impose the MA gauge condition. We perform the Cartan decomposition on the gauge fixed SU(2) link variable and then obtain the U(1) field strength and the monopole current. Finally, we measure two types of the corresponding topological charge:

- $Q_{\mathrm{SU(2)}} \equiv -\frac{1}{16\pi^2}\sum_s q(s)$

- $Q_{\mathrm{Mono}} \equiv -\frac{1}{16\pi^2}\sum_s q_{\mathrm{Mono}}(s)$

We show the scatter plots of $Q_{\mathrm{SU(2)}}$ vs. Q_{Mono} after 100 cooling sweeps in Fig.1; (a) no gauge fixing and (b) the MA gauge fixing. Obviously, there are no any correlation between two topological charges in the case of no gauge fixing, where abelian dominance is not realized. After the MA gauge fixing, a one-to-one correspondence between $Q_{\mathrm{SU(2)}}$ and Q_{Mono} reveals in the scatter plot. In further detail, there is a small variance between $Q_{\mathrm{SU(2)}}$ and Q_{Mono}. The slope of the scatter plot is not unity, but 1.34 in Fig.1 (b). In other words, it seems that the relation; $Q_{\mathrm{Mono}} \approx 0.75Q_{\mathrm{SU(2)}}$ is almost satisfied in the MA gauge on this data. To discuss the topological feature on Q_{Mono}, we show the probability distribution of Q_{Mono} in Fig.2 by using 3000 independent configurations. Several dotted lines correspond to partial contributions to the whole distribution, which are assigned to some integer value of $Q_{\mathrm{SU(2)}}$. Several discrete peeks in Fig.2

tell us that Q_{Mono} is classified by approximately discrete values. Namely, it seems that monopoles almost inherit the topological nature.

In conclusion, we have studied the topological aspects of the QCD vacuum based on the hypothesis of abelian dominance. We have found that the topological charge could be reconstructed from the monopole current and the abelian component of gauge fields if abelian dominance is realized. This indicates that the presence of monopoles preserves the non-abelian topological feature in the abelian dominated system.

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